

# Parafermionic algebras, their modules and cohomologies

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**Abstract** We explore the Fock spaces of the parafermionic algebra introduced by H.S. Green. Each parafermionic Fock space allows for a free minimal resolution by graded modules of the graded 2-step nilpotent subalgebra of the parafermionic creation operators. Such a free resolution is constructed with the help of a classical Kostant's theorem computing Lie algebra cohomologies of the nilpotent subalgebra with values in the parafermionic Fock space. The Euler-Poincaré characteristics of the parafermionic Fock space free resolution yields some interesting identities between Schur polynomials. Finally we briefly comment on parabosonic and general parastatistics Fock spaces.

## 1 Introduction

The parafermionic and parabosonic algebras were introduced by H.S. Green as a inhomogeneous cubic algebra having as quotients the fermionic and bosonic algebras with canonical (anti)commutation relations. In an attempt to find a new paradigm for quantization of classical fields H.S. Green introduced the parabosonic and parafermionic algebras encompassing the bosonic and fermionic algebras based on the canonical quantization scheme. Here we are dealing with the Fock spaces of the parafermionic algebra  $\mathfrak{g}$  of creation and annihilation operators. These Fock spaces are particular parafermionic algebra modules build at the top of a unique vacuum state by the creation operators. The creation operators close a free graded 2-step nilpotent algebra  $\mathfrak{n}$ ,  $\mathfrak{n} \subset \mathfrak{g}$ . The Fock space of a parafermionic algebra  $\mathfrak{g}$  is then defined as a quotient module of the free  $\mathfrak{n}$ -module, where the quotient ideal stems from the generalization of the Pauli exclusion principle. In this note we calculate the cohomologies  $H^\bullet(\mathfrak{n}, \mathcal{V}(p))$  of the nilpotent subalgebra  $\mathfrak{n}$  with coefficients

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in the parafermionic Fock space  $\mathcal{V}(p)$  (taken as a  $\mathfrak{n}$ -module). The cohomology ring  $H^\bullet(\mathfrak{n}, \mathcal{V}(p))$  is obtained due to by now classical Kostant's theorem [8]. With the data of  $H^\bullet(\mathfrak{n}, \mathcal{V}(p))$  one is able to construct a minimal resolution by free  $\mathfrak{n}$ -module of the Fock space  $\mathcal{V}(p)$ . Its existence is guaranteed by the Henri Cartan's results on graded algebras. It turns out that the Schur polynomials identities which have been recently put forward [9, 13] by Neli Stoilova and Joris Van der Jeugt stem from the Euler-Poincaré characteristics of the minimal free resolutions of the parafermionic and parabosonic Fock space.

## 2 Parafermionic and parabosonic algebras

The parafermionic algebra  $\mathfrak{g}$  with finite number  $n$  degrees of freedom is a Lie algebra with a Lie bracket  $[\bullet, \bullet]$  generated by the creation  $a_i^\dagger$  and annihilation  $a^j$  operators ( $i, j = 1, \dots, n$ ) having the following exchange relations

$$\begin{aligned} [[a_i^\dagger, a^j], a_k^\dagger] &= 2\delta_k^j a_i^\dagger, & [[a_i^\dagger, a^j], a^k] &= -2\delta_i^k a^j, \\ [[a_i^\dagger, a_j^\dagger], a_k^\dagger] &= 0, & [[a^i, a^j], a^k] &= 0. \end{aligned} \quad (1)$$

The parafermionic algebra  $\mathfrak{g}$  with finite number degrees of freedom  $n$  is isomorphic to the semi-simple Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha, \quad (2)$$

for a root system  $\Delta = \Delta_+ \cup \Delta_-$  of type  $B_n$  with positive roots  $\Delta_+$  given by

$$\Delta_+ = \{e_i\}_{1 \leq i \leq n} \cup \{e_i + e_j, e_i - e_j\}_{1 \leq i < j \leq n}, \quad \text{and} \quad \Delta_- = -\Delta_+.$$

Here  $\{e_i\}_{i=1}^n$  stand for the orthogonal basis in the root space,  $(e_i | e_j) = \delta_{ij}$ . One concludes that the parafermionic algebra  $\mathfrak{g}$  with  $n$  degrees of freedom is isomorphic to the orthogonal algebra  $\mathfrak{g} \cong \mathfrak{so}_{2n+1}$  endowed with the anti-involution  $\dagger$ . The physical generators correspond to the Cartan-Weyl basis  $a_i^\dagger := E^{e_i}$  and  $a^j := E^{-e_j}$ .

Similarly one defines the parabosonic algebra  $\tilde{\mathfrak{g}}$  with exchange relations (1) as the Lie super-algebra endowed with a Lie super-bracket  $[\bullet, \bullet]$  whose generators  $a_i^\dagger$  and  $a^j$  are taken to be odd generators. The parabosonic algebra  $\tilde{\mathfrak{g}}$  with  $m$  degrees of freedom is shown [3] to be isomorphic to the Lie super algebra of type  $B_{0,m}$  in the Kac table, i.e.,  $\mathfrak{osp}_{1|2m}$ . More generally, one defines the parastatistics algebra as the Lie super-algebra with  $n$  even parafermionic and  $m$  odd parabosonic degrees of freedom. The parastatistics algebra is shown to be isomorphic to the super-algebra of type  $B_{n,m}$ , i.e.,  $\mathfrak{osp}_{2n+1|2m}$  [12]. Throughout this note we will concentrate on the parafermionic algebra and its representations.

### 3 Parafermionic Fock space

The parafermionic relations (1) imply that the generators  $E_i^j = \frac{1}{2}[a_i^\dagger, a^j]$  are the matrix units satisfying

$$[E_i^j, E_k^l] = \delta_k^j E_i^l - \delta_i^l E_k^j .$$

These generators close the real form  $\mathfrak{u}$  of a linear algebra  $\mathfrak{gl}_n$  with  $(E_i^j)^\dagger = E_j^i$ .

One has decomposition of the parafermionic Lie algebra into reductive algebra  $\mathfrak{u}$  and nilpotent Lie algebras,  $\mathfrak{n}$  and  $\mathfrak{n}^*$

$$\mathfrak{g} = \mathfrak{n}^* \rtimes \mathfrak{u} \ltimes \mathfrak{n}$$

where  $\mathfrak{u}$  is the real form of the linear algebra  $\mathfrak{gl}_n$ . The free 2-step nilpotent Lie subalgebra  $\mathfrak{n} \subset \mathfrak{g}$  is generated in degree 1 by the *creation* operators  $a_i^\dagger$ ,  $V := \bigoplus_i \mathbb{C} a_i^\dagger$

$$\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 = V \oplus \wedge^2 V .$$

Analogously the annihilation operators  $a_i$  generate the subalgebra  $\mathfrak{n}^* = V^* \oplus \wedge^2 V^*$ .

The vector space  $V = \mathfrak{n}_1$  is the fundamental representation for the left action of the algebra  $\mathfrak{gl}_n$ ,  $E_i^j \cdot a_k^\dagger = \delta_k^j a_i^\dagger$ . Similarly  $V^* = \mathfrak{n}_1^*$  is the fundamental representation for the right  $\mathfrak{gl}_n$ -action,  $a^k \cdot E_i^j = \delta_i^k a^j$ . The linear algebra  $\mathfrak{gl}_n$  acts on the algebras  $\mathfrak{n}$  and  $\mathfrak{n}^*$  by automorphisms.

**Definition 1.** The parafermionic Fock space is the unitary representation  $\mathcal{V}(p)$  of the parafermionic algebra  $\mathfrak{g} \cong \mathfrak{so}_{2n+1}$  built on a unique vacuum vector  $|0\rangle$  such that

$$a_i |0\rangle = 0 , \quad [a_i, a_j^\dagger] |0\rangle = p \delta_{ij} |0\rangle . \quad (3)$$

The non-negative integer  $p$  is called the order of the parastatistics.

Let us single out a particular parabolic subalgebra  $\mathfrak{p} = \mathfrak{gl} \ltimes \mathfrak{n}$ . In the Fock representation the vacuum module  $\mathbb{C} |0\rangle$  is the trivial module for the subalgebra  $\mathfrak{p}^* = \mathfrak{n}^* \rtimes \mathfrak{gl}$ . The representation induced by  $\mathfrak{p}^*$  acting on the vacuum module is isomorphic the universal enveloping algebra of the creation algebra  $\mathfrak{n}$

$$\text{Ind}_{\mathfrak{p}^*}^{\mathfrak{g}} \mathbb{C} |0\rangle = U \mathfrak{g} \otimes_{\mathfrak{p}^*} \mathbb{C} |0\rangle \cong U \mathfrak{n} .$$

Hence the Fock representation  $\mathcal{V}(p)$  which we now describe is a particular quotient of the algebra  $U \mathfrak{n}$  created by the free action of the creation algebra  $\mathfrak{n}$ .

The  $\mathcal{V}(p)$  of parastatistics order  $p$  is a finite-dimensional  $\mathfrak{g}$ -module with a unique Lowest Weight vector  $|0\rangle$  of weight  $-\frac{p}{2} \sum_{i=1}^n e_i$  and a unique Highest Weight (HW) vector

$$|\Lambda\rangle = (a_1^\dagger)^p \dots (a_n^\dagger)^p |0\rangle \quad (4)$$

thus the  $\mathfrak{so}_{2n+1}$ -module  $\mathcal{V}(p)$  is a highest weight module of weight  $\Lambda$

$$V^\Lambda = \mathcal{V}(p) \quad \Lambda = \frac{p}{2} \sum_{i=1}^n e_i .$$

The parafermionic algebra of order  $p = 1$  coincides with the canonical fermionic Fock space, i.e., the HW representation  $\mathcal{V}(1) = V^\theta$  with  $\theta = \frac{1}{2} \sum_{i=1}^n e_i$ . The physical meaning of the order  $p$  for the parafermionic algebra is the number of particles that can occupy one and the same state, that is, we deal with a Pauli exclusion principle of order  $p$ . The symmetric submodule  $S^{p+1}\mathfrak{n}_1 \subset \mathfrak{n}_1^{\otimes p+1}$  is spanned by the “exclusion condition”  $(a_i^\dagger)^{p+1} = 0$  and it generates an ideal  $(S^{p+1}\mathfrak{n}_1)$ . The parafermionic Fock space  $\mathcal{V}(p)$  is a Lowest Weight module isomorphic to the factor module of  $U\mathfrak{n}$  by the “exclusion” ideal  $(S^{p+1}\mathfrak{n}_1)$

$$\mathcal{V}(p) \cong U\mathfrak{n} / (S^{p+1}\mathfrak{n}_1) .$$

On the other hand the parafermionic Fock space  $\mathcal{V}(p) = V^\Lambda$  is a HW  $\mathfrak{g}$ -module with HW vector  $|\Lambda\rangle$  (4)

$$V^\Lambda \cong U\mathfrak{n}^* / (S^{p+1}\mathfrak{n}_1^*) = \mathcal{V}(p) .$$

**Theorem 1 (A.J. Bracken, H.S. Green[2]).** *The HW  $\mathfrak{so}_{2n+1}$ -module  $V^\Lambda \cong \mathcal{V}(p)$  of HW vector  $|\Lambda\rangle = |p\theta\rangle$  splits into a sum of irreducible  $\mathfrak{gl}_n$ -modules  $V^\lambda$*

$$V^\Lambda \downarrow_{\mathfrak{gl}_n}^{\mathfrak{so}_{2n+1}} = \bigoplus_{\lambda: \lambda \subseteq (p^n)} V^{\lambda - (p/2)^n} , \quad \Lambda = \frac{p}{2} \sum_{i=1}^n e_i \quad (5)$$

where the sum runs over all partitions which match inside the Young diagram  $(p^n)$ .

*Proof.* The Weyl character formula applied to a Schur module  $V^\lambda$  yields the Schur polynomial

$$s_\lambda(x_1, \dots, x_n) = \sum_{w \in W_1} \varepsilon(w) e^{w(\rho_1 + \lambda)} / \sum_{w \in W_1} \varepsilon(w) e^{w(\rho_1)} \quad W_1 := S_n ,$$

where the variables are  $x_i := \exp(-e_i)$  and the vector  $\rho_1 = \frac{1}{2} \sum_{i=1}^n (n - 2i + 1) e_i$ . Alternatively the Schur polynomial is written as a quotient of determinants

$$s_\lambda(x_1, \dots, x_n) = \frac{\det \|x_j^{\rho_{1i} + \lambda_i}\|}{\det \|x_j^{\rho_{1i}}\|} . \quad (6)$$

The Weyl character formula applied to the  $\mathfrak{so}_{2n+1}$ -module  $V^\Lambda$  reads

$$\chi^\Lambda = D_{\rho+p\theta} / D_\rho = e^{p\theta} \sum_{\lambda: l(\lambda') \leq p} s_\lambda(x_1, \dots, x_n) , \quad e^{p\theta} = (x_1 \dots x_n)^{-\frac{p}{2}} \quad (7)$$

where  $W = S_n \ltimes \mathbb{Z}_2^n$  is the Weyl group of the root system of Dynkin type  $B_n$  and  $D_\rho = \sum_{w \in W} \varepsilon(w) e^{w\rho}$  with  $\rho = \frac{1}{2} \sum_{i=1}^n (2n - 2i + 1) e_i$ . The quotient of determinants  $D_{\rho+p\theta} / D_\rho$  can be further expanded as a sum over the Schur polynomials with no more than  $p$  columns (see p.84 in the book of Macdonald [11]). Here  $\lambda'$  stands for the partition conjugated to  $\lambda$  and  $l(\mu)$  is the length of the partition  $\mu$ . The Schur polynomials  $s_\lambda(x)$  are characters of the  $\mathfrak{gl}_n$ -modules thus the expansion of the  $\mathfrak{so}_{2n+1}$ -character  $\chi^\Lambda$  implies the branching formula (5). We are done.  $\square$

#### 4 Kostant's theorem and the cohomology $H^\bullet(\mathfrak{n}, \mathcal{V}(p))$

The Kostant theorem is a powerful tool helping to calculate cohomologies. Let's have a semi-simple algebra  $\mathfrak{g}$  and its Borel subalgebra  $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$ . Any parabolic subalgebra  $\mathfrak{p}$ ,  $\mathfrak{g} \supset \mathfrak{p} \supseteq \mathfrak{b}$  has a Levi decomposition  $\mathfrak{p} = \mathfrak{g}_1 \ltimes \mathfrak{n}$  where  $\mathfrak{g}_1$  is a reductive algebra and  $\mathfrak{n}$  is the nilradical (largest nilpotent ideal) of  $\mathfrak{p}$ . Consider the  $\mathfrak{g}$ -module  $V^\Lambda$  of weight  $\Lambda$  and the cohomology  $H^\bullet(\mathfrak{n}, V^\Lambda)$  with coefficients in the restriction  $\mathfrak{n}$ -module  $V^\Lambda \downarrow_{\mathfrak{n}}^{\mathfrak{g}}$ . The Kostant's theorem gives the decomposition of  $H^\bullet(\mathfrak{n}, V^\Lambda)$  as a sum of irreducibles  $\mathfrak{g}_1$ -modules  $V^\mu$ .

**Theorem 2.** (Kostant) *Let  $W$  be the Weyl group of the algebra  $\mathfrak{g}$  and the subset  $\Phi_\sigma \subseteq \Delta_+$  be*

$$\Phi_\sigma := \sigma\Delta_- \cap \Delta_+ \subseteq \Delta_+.$$

*Let  $\rho$  be the Weyl vector  $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$ . The roots of the nilpotent radical  $\mathfrak{n}$  are denoted as  $\Delta(\mathfrak{n})$  and the subset  $W^1 = \{\sigma \in W \mid \Phi_\sigma \subset \Delta(\mathfrak{n})\}$  is a cross section of the coset  $W_1 \backslash W$ . The cohomology  $H^\bullet(\mathfrak{n}, V^\Lambda)$  has a decomposition into irreducible  $\mathfrak{g}_1$ -modules  $V^\mu$*

$$H^\bullet(\mathfrak{n}, V^\Lambda) = \bigoplus_{\sigma \in W^1} V^{\sigma(\rho + \Lambda) - \rho}$$

*where the cohomological degree of  $H^j(\mathfrak{n})$  is the number of the elements  $j := \#\Phi_\sigma$ .*

J. Grassberger, A. King and P. Tiraó [4] applied Kostant's theorem to cohomology  $H^\bullet(\mathfrak{n}, \mathbb{C})$  with trivial coefficients. Here we extend their method for cohomologies with coefficients in the parafermionic Fock space  $\mathcal{V}(p)$ ,  $H^\bullet(\mathfrak{n}, \mathcal{V}(p))$ .

**Theorem 3.** *Let  $\mathfrak{n}$  be the free 2-step nilpotent Lie algebra  $\mathfrak{n} = V \oplus \wedge^2 V$  and  $V^\Lambda$  be the parafermionic Fock space,  $V^\Lambda = \mathcal{V}(p)$ . The cohomology  $H^\bullet(\mathfrak{n}, V^\Lambda)$  with values in the  $\mathfrak{n}$ -module  $V^\Lambda \downarrow_{\mathfrak{n}}^{\mathfrak{g}}$  has a decomposition into irreducible  $\mathfrak{gl}(V)$ -modules*

$$H^k(\mathfrak{n}, \mathcal{V}(p)) \cong \bigoplus_{\mu: \mu = \mu'} V^{*\mu^{(p)} - (\frac{p}{2})^n}, \quad k = \frac{1}{2}(|\mu| + r(\mu)), \quad (8)$$

*where the sum is over self-conjugated Young diagrams  $\mu = (\alpha|\alpha)$  and the notation  $\mu^{(p)}$  stays for the  $p$ -augmented diagram  $\mu^{(p)} = (\alpha + p|\alpha)$ .*

We recall the Frobenius notation for a Young diagram  $\eta$

$$\eta := (\alpha_1, \dots, \alpha_r | \beta_1, \dots, \beta_r) \quad r = r(\eta)$$

where the *rank*  $r(\eta)$  is the number of boxes on the diagonal of  $\eta$ , the arm-length  $\alpha_i$  is the number of boxes on the right of the  $i$ th diagonal box, and the leg-length  $\beta_i$  is the number of boxes below the  $i$ th diagonal box. The overall number of boxes in  $\eta$  is  $|\eta| = r + \sum_{i=1}^r \alpha_i + \sum_{i=1}^r \beta_i$ . The conjugated diagram  $\eta'$  is the diagram in which the arms and legs are exchanged

$$\eta' := (\beta_1, \dots, \beta_r | \alpha_1, \dots, \alpha_r).$$

*Proof.* The parafermionic algebra  $\mathfrak{g} \cong \mathfrak{so}_{2n+1}$  has Cartan decomposition (2). Consider its parabolic subalgebra  $\mathfrak{p} = \bigoplus_{i>j} \mathfrak{g}_{e_i-e_j} \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \subset \mathfrak{g}$ . From the parafermionic relations (1) is readily seen that the Levi decomposition of the parabolic subalgebra  $\mathfrak{p} = \mathfrak{g}_1 \ltimes \mathfrak{n}$  has reductive component

$$\mathfrak{g}_1 = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{e_i-e_j} \cong \mathfrak{gl}_n \quad (9)$$

acting by automorphisms on the free 2-step nilpotent algebra  $\mathfrak{n}$  (the space  $\mathfrak{n}_1 = V$  being the fundamental representation of  $\mathfrak{g}_1 = \mathfrak{gl}_n$ )

$$\mathfrak{n} = \bigoplus_i \mathfrak{g}_{e_i} \oplus \bigoplus_{i<j} \mathfrak{g}_{e_i+e_j} \cong V \oplus \wedge^2 V. \quad (10)$$

The Weyl group  $W_1$  of  $\mathfrak{g}_1 = \mathfrak{gl}_n$  is the symmetric group  $S_n$  operating on  $\{e_1, \dots, e_n\}$  by permutations. The Weyl group of  $\mathfrak{g} = \mathfrak{so}_{2n+1}$  is  $W = S_n \ltimes \mathbb{Z}_2^n$ . The  $\mathbb{Z}_2^n$  is generated by operators  $\tau_i, i = 1, \dots, n$  such that  $\tau_i^2 = 1$  acting by

$$\tau_i(e_j) = \begin{cases} -e_j & i = j \\ e_j & i \neq j \end{cases}.$$

The elements  $\tau_I \in \mathbb{Z}_2^n$  are indexed by subsets  $I \subseteq \{1, \dots, n\}$ ,  $\tau_I \in \prod_{i \in I} \tau_i$ .

Let us describe the subset  $W^1$  which has order  $|W^1| = 2^n$ . Both  $W^1$  and  $\mathbb{Z}_2^n$  are cross sections of  $W_1 \backslash W$  thus for each  $\tau_I \in \mathbb{Z}_2^n$  exists a unique permutation  $\omega_I \in S_n$  such that  $\omega_I \tau_I \in W^1$ .

Let  $\mathfrak{b}^0$  be the nilpotent part of the Borel algebra  $\mathfrak{b}^0 = \mathfrak{b}/\mathfrak{h}$  and the complement be  $\mathfrak{m}_1 = \mathfrak{g}_1 \cap \mathfrak{b}^0 = \mathfrak{b}^0/\mathfrak{n}$ . The subset  $W^1 = \{\sigma \in W | \Phi_\sigma \subseteq \Delta(\mathfrak{n})\}$  keeps stable also the complement of  $\Delta(\mathfrak{n})$

$$\sigma \Delta(\mathfrak{n}) \subseteq \Delta_+ \quad \Leftrightarrow \quad \sigma^{-1} \Delta(\mathfrak{b}^0/\mathfrak{n}) \subseteq \Delta_+.$$

The root system of  $\mathfrak{m}_1$  is  $\Delta(\mathfrak{m}_1) = \{e_i - e_j, i < j\}$  therefore  $\omega_I \tau_I \in W^1$  implies  $\tau_I^{-1} \omega_I^{-1} \Delta(\mathfrak{m}_1) \subseteq \Delta_+$  or  $\tau_I \omega_I^{-1}(e_i - e_j) > 0$  for  $i < j$ . These inequalities are satisfied for  $\omega_I \in S_n$  defined by

$$\omega_I(a) > \omega_I(b) \quad \text{when} \quad \begin{cases} a < b & a \in I, b \in I \\ a > b & a \notin I, b \notin I \\ & a \in I, b \notin I \end{cases}.$$

The permutation places all elements of  $I = \{i_1, \dots, i_r\}$  after all the elements of its complement  $\bar{I}$  preserving the order of  $\bar{I}$  and reversing the order of  $I$ , that is,

$$\omega_I(1, \dots, i_1, \dots, i_r, \dots, n) = (1, \dots, \hat{i}_1, \dots, \hat{i}_r, \dots, n, i_r, \dots, i_2, i_1). \quad (11)$$

The permutation  $\omega_I$  can be represented as a product of cyclic permutations  $\omega_I = \zeta_{i_r} \dots \zeta_{i_2} \zeta_{i_1}$  where  $\zeta_{i_k}$  is the cycle (of length  $n - i_k + 1$ ) from positions  $i_k - k + 1$  to  $n - k + 1$ . Therefore the action of  $\omega_I$  is represented by the sequence of steps

$$\begin{aligned}
\zeta_{i_1}(1, \dots, i_1, \dots, i_k, \dots, n) &= (1, \dots, \hat{i}_1, i_1 + 1, \dots, n, i_1), \\
\zeta_{i_2}(1, \dots, \underbrace{i_2}_{\text{place } i_2-1}, \dots, n, i_1) &= (1, \dots, \hat{i}_2, \dots, n, i_2, i_1), \\
&\dots \\
\zeta_{i_k}(1, \dots, \underbrace{i_k}_{\text{place } i_k-k+1}, \dots, n, i_{k-1}, \dots, i_1) &= (1, \dots, \hat{i}_k, \dots, n, i_k, \dots, i_1).
\end{aligned}$$

Note that after the  $j$ -th step, the last  $j$  places are not touched by the next cyclings.

The Weyl vector  $\rho$  associated to  $\mathfrak{g} = \mathfrak{so}_{2n+1}$  reads  $\rho = \frac{1}{2} \sum_{i=1}^n (2n - 2i + 1)e_i$ . Note that the components of  $\rho$  are strictly decreasing with step  $1 = \rho_{i+1} - \rho_i$ . The cohomology ring  $H^\bullet(\mathfrak{n}, V^\Lambda)$  decomposes into  $\mathfrak{gl}(V)$ -modules with HW weights  $\sigma(\rho + \Lambda) - \rho$  for  $\sigma \in W^1$ . We are interested in the case  $\Lambda = \frac{p}{2} \sum e_i$ ,  $V^\Lambda = \mathcal{V}(p)$ .

Consider first the case  $p = 0$ , i.e., the cohomology with trivial coefficients  $H^\bullet(\mathfrak{n}, \mathbb{C})$  following [4]. The highest weights  $\lambda_I = \sigma(\rho) - \rho$  for  $\sigma \in W^1$  are non-positive due to  $\sigma(\rho)_i \leq \rho_i$ . The cycling structure of  $\omega_I$  implies

$$\lambda_I = \sum \lambda_j e_j, \quad \lambda_j = -(n - i_{n-j+1} + 1) \chi_{(n-r+1 \leq j \leq n)} - \sum_{k=1}^r \chi_{(i_k - k + 1 \leq j \leq n - k)}.$$

One has an isomorphism between a HW  $\mathfrak{gl}_n$ -module  $V^{\lambda_I}$  with negative weight  $\lambda_I \leq 0$  and the dual representation  $V^{*\mu_I}$  with reflected weight  $\mu_I \geq 0$

$$V^{\lambda_I} \cong V^{*\mu_I} \quad \mu_I := \sum_{i=1}^n \mu_i e_i = - \sum_{i=1}^n \lambda_{n-i+1} e_i \geq 0.$$

The components of  $\mu_I$  are decreasing positive integers  $\mu_1 \geq \dots \geq \mu_n \geq 0$

$$\mu_j = (n - i_j + 1) \chi_{(1 \leq j \leq r)} + \sum_{k=1}^r \chi_{(k+1 \leq j \leq n - i_k + k)}, \quad (12)$$

and these components code a self-conjugated Young diagram  $\mu'_I = \mu_I$

$$\mu_I = (\alpha_I | \alpha_I) \quad \alpha_I = (\alpha_1, \dots, \alpha_r), \quad \text{for } \alpha_j = n - i_j.$$

Roughly speaking the  $j$ -th cyclic permutation  $\zeta_{i_k}$  in  $\omega_I$  creates a self-conjugated hook subdiagram of  $\mu_I$  with  $\alpha_j = n - i_j$ .

By virtue of the Kostant's theorem [8] the cohomology  $H^\bullet(\mathfrak{n}, \mathbb{C})$  of the nilpotent Lie algebra  $\mathfrak{n}$  has decomposition into Schur modules with HW vector  $|\mu_I\rangle$

$$H^\bullet(\mathfrak{n}, \mathbb{C}) = \bigoplus_{\mu_I: \mu'_I = \mu_I} V^{*\mu_I}, \quad |\mu_I\rangle = E^{-\Phi_\sigma}, \quad \sigma \in W^1$$

labelled by self-conjugated Young diagrams. All self-conjugated Young diagrams  $\{\mu_I : \mu'_I = \mu_I\}$  are in bijection with elements of  $W^1$  (with cardinality  $|W^1| = 2^n$ ), all these diagrams are included into the maximal square diagram,  $\mu_I \subseteq (n^n)$ .

Consider now the cohomology ring  $H^\bullet(\mathfrak{n}, V^\Lambda)$  where  $\Lambda = \frac{p}{2} \sum e_i$ . It decomposes into  $\mathfrak{gl}_n$ -modules with HW weights  $\lambda_I^{(p)} = \sigma(\rho + \Lambda) - \rho$  where  $\sigma = \omega_I \tau_I \in W^1$ . Given a set  $I = \{i_1, \dots, i_r\}$  the shift  $\Lambda$  modifies the dominant weight  $v_I = \sum v_i e_i$  to

$$v_j^{(p)} = -\lambda_{n-j+1}^{(p)}, \quad v_j^{(p)} = -\frac{p}{2} + (n - i_j + 1 + p) \chi_{(1 \leq j \leq r)} + \sum_{k=1}^r \chi_{(k+1 \leq j \leq n-i_k+k)}.$$

The weights  $v_I^{(p)} = \mu_I^{(p)} - \frac{p}{2} \sum e_i$  fix the HW vectors in the  $\mathfrak{gl}_n$ -modules  $V^* v_I^{(p)}$

$$V^* v_I^{(p)} = V^* \mu_I^{(p)} \otimes |\Lambda\rangle \quad \text{where} \quad \mu_I^{(p)} = (\alpha_I + p|\alpha_I) \quad \alpha_j = n - i_j$$

from where the decomposition of  $H^\bullet(\mathfrak{n}, \mathcal{V}(p))$  (8) follows, the sum over  $\sigma \in W^1$  in Kostant's theorem being replaced by the sum over self-conjugated Young diagrams  $\mu = \mu'$ . The arm  $p$ -augmented diagram  $\mu_I^{(p)}$  stems from the self-conjugated diagram  $\mu_I = (\alpha_I | \alpha_I)$  cf. eq. (12) by augmenting the arm-lengths,  $\mu_I^{(p)} = (\alpha_I + p|\alpha_I)$ .

The cohomological degree  $k$  of the elements in  $V^* \mu_I^{(p)} \otimes |0\rangle \subset H^k(\mathfrak{n}, \mathcal{V}(p))$  do not depend on  $p$  but only on  $\sigma = \omega_I \tau_I \in W^1$  (or equivalently on  $\mu_I$ ). In view of  $\Phi_\sigma = \Delta_- \cap \sigma^{-1} \Delta_+$  a root  $\xi \in \Phi_\sigma \subseteq \Delta(\mathfrak{n})$  whenever  $\sigma^{-1} \xi < 0$ . But the set  $\Delta(\mathfrak{n})$  is stable under permutations and  $\tau_I^{-1} = \tau_I$  thus

$$\begin{aligned} \#\Phi_\sigma &= \#\{\xi \in \Delta(\mathfrak{n}), \tau_I \xi < 0\} \\ &= \#\{\mathfrak{g}_{e_i}, i \in I\} + \#\{\mathfrak{g}_{e_i+e_j} : i < j, i \in I\} \\ &= \sum_{i \in I} (1 + n - i) = r + \sum_{k=1}^r (n - i_k) = r + s = \deg \mu_I. \end{aligned}$$

Thus the cohomological degree  $k = \deg \mu_I = \#\Phi_\sigma$  is the total degree  $k = (r + s)$  of the bi-complex  $\wedge^s(\wedge^2 V^*) \otimes \wedge^s V^*$ . The number of boxes above the diagonal in  $\mu_I$  is  $s = \frac{1}{2}(|\mu_I| - r)$  so finally one gets  $k = \deg \mu_I = \frac{1}{2}(r(\mu_I) + |\mu_I|)$ . We are done.  $\square$

## 5 Resolution of $\mathcal{V}(p)$

A general result of Henri Cartan [1] states that every positively graded  $\mathcal{A}$ -module  $M$  of a graded algebra  $\mathcal{A} = \oplus_{n \geq 0} \mathcal{A}_n$  allows for a minimal projective resolution by projective  $\mathcal{A}$ -modules. Moreover the notions of a projective and a free module coincide in the graded category. Thus for every positively graded  $\mathcal{A}$ -module  $M$  there exists a minimal resolution by free  $\mathcal{A}$ -modules.

The universal enveloping algebra  $U\mathfrak{n}$  is a graded associative algebra and the parafermionic Fock space  $\mathcal{V}(p) = V^\Lambda$  is a positively graded  $U\mathfrak{n}$ -module. There exists [1] a minimal free resolution  $P_\bullet = \bigoplus_{k=0}^N P_k$  of the right  $U\mathfrak{n}$ -module  $\mathcal{V}(p)^*$

$$0 \rightarrow P_N \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathcal{V}(p)^* \rightarrow 0 \quad (13)$$



by free right  $U\mathfrak{n}$ -modules  $P_k = E_k \otimes U\mathfrak{n}$ . We apply the functor  $- \otimes_{U\mathfrak{n}} \mathbb{C}$  on the complex  $P_\bullet$ , where  $\mathbb{C}$  is the trivial  $U\mathfrak{n}$ -module. The minimality of the resolution  $P_\bullet$  implies [1] that the differentials of the complex  $P_\bullet \otimes_{U\mathfrak{n}} \mathbb{C}$  vanish. Hence the multiplicity spaces  $E_k$  coincide with the homologies

$$E_k \cong \text{Tor}_k^{U\mathfrak{n}}(\mathcal{V}(p)^*, \mathbb{C}) = H_k(\mathfrak{n}, \mathcal{V}(p)^*) \quad \Rightarrow \quad E_k^* \cong H^k(\mathfrak{n}, \mathcal{V}(p)) ,$$

where we used the isomorphism  $H_k(\mathfrak{n}, M)^* = H^k(\mathfrak{n}, M^*)$ . Theorem 3 gives us the spaces  $E_k \cong H^k(\mathfrak{n}, \mathcal{V}(p))^*$  so we have constructed the minimal free resolution (13).

**Theorem 4.** *The Euler-Poincaré characteristic of the free minimal resolution of the (dual of the) parafermionic Fock space  $\mathcal{V}(p)$  (13) yields the identity*

$$\frac{\sum_{\mu: \mu=\mu'} (-1)^{\frac{1}{2}(|\mu|+r(\mu))} s_{\mu(p)}(x)}{\prod_i (1-x_i) \prod_{i < j} (1-x_i x_j)} = \sum_{\lambda: l(\lambda') \leq p} s_\lambda(x) . \quad (14)$$

*Proof.* In general, the mapping of modules of an algebra into its Grothendieck ring of characters is an example of Poincaré-Euler characteristic. The free resolution (13) is naturally a (reducible)  $\mathfrak{gl}(V)$ -module and the Schur functions (6) span the ring of  $\mathfrak{gl}(V)$ -characters. All the homology of a resolution is concentrated in degree 0, hence on the RHS of (14) stays the character of the self-conjugated<sup>1</sup> module  $\mathcal{V}(p)$ (7)

$$\text{ch} \mathcal{V}(p) = \text{ch} \mathcal{V}(p)^* = e^{-p\theta} \sum_{\lambda \subseteq (p^n)} s_\lambda(x) \quad x_i := \exp(e_i) .$$

From the Poincaré-Birkhoff-Witt theorem follows that the character of  $P_k$  reads

$$\text{ch} P_k = \text{ch}(E_k \otimes U\mathfrak{n}) = \frac{e^{-\Lambda} s_{\mu(p)}(x)}{\prod_i (1-x_i) \prod_{i < j} (1-x_i x_j)} .$$

Thus the alternating sum on the LHS comes from the characters of the  $\mathfrak{gl}(V)$ -modules  $E_k \otimes U\mathfrak{n}$  taken with alternating signs corresponding to the homological degree. The factor  $e^{p\theta} = e^\Lambda$  accounting for the weight of the HW vector  $|\Lambda\rangle$  cancels which proves the parafermionic sign-alternating identity (14).  $\square$

**Remark.** The free minimal resolution of the trivial module  $\mathbb{C}$  constructed by Józefiak and Weyman [6] with the help of the homologies  $H_k(\mathfrak{n}, \mathbb{C})$  corresponds to the resolution  $P_\bullet$  (13) of  $\mathbb{C} \cong \mathcal{V}(p=0)$ .

The parafermionic sign-alternating identity (14) was proposed by Stoilova and Van der Jeugt in their study of parafermionic Fock space [13]. The parabosonic Fock space has been explored in [9] where the “super-symmetric partner” of the identities (14) has been proposed (for a combinatorial proof see [7])

$$\frac{\sum_{\mu: \mu=\mu'} (-1)^{\frac{1}{2}(|\mu|+r(\mu))} s_{[\mu(p)]'}(x)}{\prod_i (1-x_i) \prod_{i < j} (1-x_i x_j)} = \sum_{\lambda: l(\lambda) \leq p} s_\lambda(x) . \quad (15)$$

<sup>1</sup> The self-conjugacy  $\mathcal{V}(p) \cong \mathcal{V}(p)^*$  allows to switch between  $x_i := \exp(\pm e_i)$  without a conflict.

The parity functor  $\Pi$  switches parafermionic *even* generators to parabosonic *odd* generators, thus  $\mathfrak{g} = \mathfrak{so}_{2n+1} \xrightarrow{\Pi} \tilde{\mathfrak{g}} = \mathfrak{osp}_{1|2n}$ . The effect of  $\Pi$  is the passage  $\lambda \xrightarrow{\Pi} \lambda'$ . The identity (15) is rooted into a minimal free resolution of the parabosonic Fock space  $\tilde{\mathcal{V}}(p) = \Pi \mathcal{V}(p)$  by free  $U\tilde{\mathfrak{n}}$ -modules of the nilpotent Lie super-algebra  $\tilde{\mathfrak{n}} \subset \tilde{\mathfrak{g}}$ .

More generally, one can consider the parastatistics Fock space  $\mathcal{V}_{n|m}(p)$  of the parastatistics Lie super-algebra  $\mathfrak{g}_{n|m} := \mathfrak{osp}_{2n+1|2m}$  with  $n$  parafermionic and  $m$  parabosonic modes. We conjecture that there exists a complex of free  $U\mathfrak{n}_{n|m}$ -modules of the maximal nilpotent Lie superalgebra  $\mathfrak{n}_{n|m} \subset \mathfrak{osp}_{2n+1|2m}$  whose cohomology is  $\mathcal{V}_{n|m}(p)$ . Then the Euler-Poincare characteristics of such a complex will yield one more identity (which was obtained by different method in [10])

$$\frac{\prod_{i < j, i \neq j} (1 + x_i x_j) \sum_{\mu: \mu = \mu'} (-1)^{\frac{1}{2}(|\mu| + r(\mu))} h_{s_{\mu(p)}}(x)}{\prod_i (1 - x_i) \prod_{i < j, i \neq j} (1 - x_i x_j)} = \sum_{\lambda: \lambda_1 \leq p} h_{s_{\lambda}}(x) .$$

Here the  $(n|m)$ -hook Schur polynomial  $h_{s_{\lambda}}(x)$  is the character of the irreducible  $\mathfrak{gl}_{n|m}$ -module  $V^{\lambda}$ ,  $h_{s_{\lambda}}(x) = ch V^{\lambda}$ . The non-trivial  $\mathfrak{gl}_{n|m}$ -modules  $V^{\lambda}$  are labelled by diagrams  $\lambda$  such that  $\lambda_{n+1} \leq m$ .

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